Advanced II	September 23, 2025

Lecture 5: Linear Programming: Duality

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In this lecture we first discuss integrality of the vertex cover LP and then turn to duality in linear programming:

- We introduce duality of linear programming.
- We prove weak-duality and state strong-duality.
- We then see how strong-duality implies complementarity slackness.

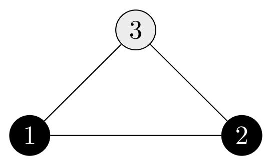
These notes are based on [1].

1 Vertex Cover

The definition of the vertex cover problem is:

Definition 1 *Given a graph $G = (V, E)$ with node-weights $w : V \to \mathbb{R}$ find a vertex cover C (i.e., $e \cap C \neq \emptyset$ for all $e \in E$) that minimizes $w(C) = \sum_{v \in C} w(v)$.*

Example 1 *The minimum weight vertex cover (depicted in black) of the following graph is 3 (obtained by taking the vertices of weight 1 and 2).*



Similarly to the case of maximum weight bipartite perfect matching, we formulate a linear program for the vertex cover problem. Here, we have a variable x_v for each vertex v with the intended meaning that it takes value 1 if that vertex is in the vertex cover. The constraints say that each edge needs to be covered. The LP can be formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{v \in V} x_v w(v) \\ \text{Subject to} \quad & x_u + x_v \geq 1 \quad \forall \{u, v\} \in E \\ & 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned}$$

We have the following structural result for bipartite graphs:

Claim 2 *For bipartite graphs, any extreme point to the vertex cover LP is integral.*

Proof Consider an extreme point x^* and let $V_f = \{v : 0 < x^*_v < 1\}$ be those vertices with fractional values in x^*. Suppose toward contradiction that $V_f \neq \emptyset$. Further, let A, B be the bipartition of V and let $A_f = V_f \cap A$ and $B_f \cap V_f$ be the fractional vertices in A and B, respectively. As for bipartite matchings, we reach a contradiction by defining feasible solutions y and z so that $x^* = \frac{1}{2}(y + z)$ which contradicts that x^* is an extreme point. Let $\epsilon = \min\{x_v, (1 - x_v) : v \in A_f \cup B_f\}$ and define y and z by

$$y_v = \begin{cases} x^*_v + \epsilon & \text{if } v \in A_f \\ x^*_v - \epsilon & \text{if } v \in B_f \\ x^*_v & \text{otherwise} \end{cases}$$

[1]**Disclaimer:** These notes were written as notes for the lecturer. They have not been peer-reviewed and may contain inconsistent notation, typos, and omit citations of relevant works.

$$z_v = \begin{cases} x_v^* - \epsilon & \text{if } v \in A_f \\ x_v^* + \epsilon & \text{if } e \in B_f \\ x_v^* & \text{otherwise} \end{cases}$$

Clearly, we have that $x^* = \frac{1}{2}(y+z)$. It remains to verify that y and z are feasible solutions. Let us verify y (the argument that z is feasible is the same). By the selection of ϵ, we have that $0 \le y_v \le 1$ for all $v \in V$. We now need to verify that the constraint for every edge is satisfied. Consider an edge $\{a,b\} \in E$ in the bipartite graph. We need to verify that $y_a + y_b \ge 1$. If $x_a = 1$ (or $x_b = 1$), then we have that $y_a = 1$ (or $y_b = 1$) and so the constraint holds. Otherwise $0 < x_a, x_b < 1$ and so $a \in A_f$ and $b \in B_f$. This in turn implies that y is feasible since

$$y_a + y_b = (x_a + \epsilon) + (x_b - \epsilon) = x_a + x_b \ge 1\,,$$

where the last inequality holds because x^* is a feasible solution. ■

The above structural result says that we can solve minimum weighted vertex cover on bipartite graphs by simply solving the above linear program to find an optimal extreme point. It does not work for general graphs which can be seen by considering the triangle (as for matchings). In contrast to matchings, the vertex cover problem on general graphs is an NP-hard problem and we do not expect to have efficient algorithms for it.

2 Linear Programming Duality

2.1 Intuition

Consider the following linear program:

$$\begin{aligned} \textbf{Minimize} \quad & 7x_1 + 3x_2 \\ \textbf{Subject to:} \quad & x_1 + x_2 \ge 2, \quad / \cdot y_1 \\ & 3x_1 + x_2 \ge 4, \quad / \cdot y_2 \\ & x_1, x_2 \ge 0. \end{aligned}$$

We are looking for the optimal solution OPT to this LP. To find the solution we may ask two types of questions (to find the upper and lower bound on OPT). Is there a solution of cost ≤ 10? (Is OPT ≤ 10)? An answer to this type of questions is quite simple; we just find a feasible solution to the LP with objective function ≤ 10 e.g. $x_1 = x_2 = 1$. Is there no better solution? (Is OPT ≥ 10)? Observe that we can bound the objective function value using constraints that every feasible solution satisfies. From the first constraint we get

$$7x_1 + 3x_2 \ge x_1 + x_2 \ge 2.$$

Thus OPT ≥ 2. Similarly from the second constraint we get OPT ≥ 4. To make a better lower bound we will need to be more clever. Let's take a linear combination of the constraints with coefficients y_1, y_2 correspondingly. y_1, y_2 should be non-negative because multiplying a constraint by negative number would flip the inequality. By taking $y_1 = 1,\ y_2 = 2$ we obtain

$$7x_1 + 3x_2 \ge (x_1 + x_2) + 2(3x_1 + x_2) \ge 2 + 2 \cdot 4 = 10.$$

For each constraint of the given LP we associate a dual variable y_i denoting the weight of the i-th constraint. What kind of variables can we take to get a valid lower bound? How should we pick coefficients to maximize lower bound for OPT? First of all we are interested in lower-bounding the objective function. Thus linear combination of primal constraints cannot exceed primal objective function. As mentioned earlier $y_i \geq 0$. In this way we get the following (dual) linear program for y_1, y_2:

$$
\begin{aligned}
\textbf{Maximize}\ & 2y_1 + 4y_2 && \text{(lower bound as tight as possible)} \\
\textbf{Subject to:}\ & y_1 + 3y_2 \leq 7, && \text{(coefficient of } x_1\text{)} \\
& y_1 + y_2 \leq 3, && \text{(coefficient of } x_2\text{)} \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Let's try now to formalize this approach.

2.2 General case

Consider the following linear program with n variables x_i for $i \in [1, n]$ and m constraints:

$$
\begin{aligned}
\textbf{Minimize}\ & \sum_{i=1}^{n} c_i x_i \\
\textbf{Subject to:}\ & \sum_{i=1}^{n} A_{ji} x_i \geq b_j \quad \forall j = 1, \ldots, m, \\
& x \geq 0.
\end{aligned}
$$

Then, the dual program has m variables y_j for $j \in [1, m]$ and n constraints:

$$
\begin{aligned}
\textbf{Maximize}\ & \sum_{j=1}^{m} b_j y_j \\
\textbf{Subject to:}\ & \sum_{j=1}^{m} A_{ji} y_j \leq c_i \quad \forall i = 1, \ldots, n, \\
& y \geq 0.
\end{aligned}
$$

Each variable y_j in the dual program corresponds to the weight of one of constraints from the primal LP. We have n constraints in the dual, one for every primal variable x_i.
Remark We showed how to produce dual program for minimization problem. Similar approach works also for maximization problems. Also every maximization problem can be reduced to a minimization one. We replace x_i with $-x_i$ in constraints and objective function obtaining minimization problem.

Remark One can verify that if we take the dual of the dual problem, we get back to the primal problem, as we should expect. Finding the dual linear program is an automatic procedure.

2.3 Duality Theorems

Let's focus now on the solutions of both LPs. In our example optimal solutions to primal and dual problems coincided. We now present two theorems that connect primal and dual solutions.

Theorem 3 (Weak Duality) *If x is* primal-feasible *(meaning that x is a feasible solution to the primal problem) and y is* dual-feasible*, then*

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{j=1}^{m} b_j y_j.$$

Proof Let's rewrite the right hand side

$$\sum_{j=1}^{m} b_j y_j \leq \sum_{j=1}^{m}\sum_{i=1}^{n} A_{ji} x_i y_j = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} A_{ji} y_j\right) x_i \leq \sum_{i=1}^{n} c_i x_i.$$

Here we used the fact that $x, y \geq 0$ for the inqualities.
■

This theorem tells us that every dual-feasible solution is a lower bound to any primal solution. This is intuitive: every primal feasible solution satisfies primal constraints, dual feasible solution gives us a way of lower bounding primal solution using primal constraints. Moreover from Weak Duality we can conclude that optimal solution to primal program is lower bounded by optimal solution to dual program. In fact optimal solutions to primal and duals linear programs coincide, leading to the following theorem.

Theorem 4 (Strong Duality) *If x is an optimal primal solution and y is an optimal dual solution, then*

$$\sum_{i=1}^{n} c_i x_i = \sum_{j=1}^{m} b_j y_j.$$

Furthermore, if the primal problem is unbounded, then the dual problem is infeasible and analogously if the dual is unbounded, the primal is infeasible.

We omit the proof of strong duality for now but I encourage you to understand one of the many proofs!

2.4 Example: Maximum cardinality matching and Vertex Cover on Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph and let M be a matching. Let x_e be a variable corresponding to taking edge $e \in M$. We want to maximize the cardinality of M while assuring that every vertex has at most one neighboring edge belonging to M. Writing those conditions in a form of LP gives us:

$$\begin{aligned} \textbf{Maximize} \quad & \sum_{e \in E} x_e \\ \textbf{Subject to:} \quad & \sum_{e=(a,b)\in E} x_e \leq 1 \quad \forall a \in A, \\ & \sum_{e=(a,b)\in E} x_e \leq 1 \quad \forall b \in B, \\ & x_e \geq 0. \end{aligned}$$

Thus the dual program looks like this:

$$\begin{aligned} \textbf{Minimize} \quad & \sum_{v \in A\cup B} y_v \\ \textbf{Subject to:} \quad & y_a + y_b \geq 1 \quad \text{for } (a,b) \in E, \\ & y_v \geq 0. \end{aligned}$$

One can easily notice that this LP is vertex cover relaxation. By weak-duality, we have that $|M| \leq |C|$ for any matching M and vertex cover C. Moreover, since both the primal and the dual are integral for bipartite graphs, strong LP-duality implies König's theorem:

Theorem 5 (König 1931) *Let $M^\star$ be a maximum cardinality matching and $C^\star$ be a minimum vertex cover of a bipartite graph. Then*

$$|M^\star| = |C^\star|.$$

Another well-known duality that is also a special case of LP-duality is the max-flow= min-cut theorem.

2.5 Complementarity Slackness

Strong duality gives an important relationship between primal and dual optimal solutions.

Theorem 6 *Let $x \in \mathbb{R}^n$ be a feasible solution to the primal and let $y \in \mathbb{R}^m$ be a feasible solution to the dual. Then*

$$x, y \text{ are both optimal solutions} \iff \begin{cases} x_i > 0 \Rightarrow c_i = \sum_{j=1}^{m} A_{ji} y_j & \forall i = 1, \ldots, n, \\ y_j > 0 \Rightarrow b_j = \sum_{i=1}^{n} A_{ji} x_i & \forall j = 1, \ldots, m. \end{cases}$$

Proof We will apply the strong duality theorem to the weak duality theorem proof.

⇒ Let x be the optimal primal solution. From the weak duality theorem proof, we have that

$$\sum_{j=1}^{m} b_j y_j \leq \sum_{j=1}^{m} \sum_{i=1}^{n} A_{ji} x_i y_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{ji} y_j \right) x_i \leq \sum_{i=1}^{n} c_i x_i. \tag{1}$$

Here we used the fact that $x, y \geq 0$. On the other hand by the strong duality theorem

$$\sum_{j=1}^{m} b_j y_j = \sum_{i=1}^{n} c_i x_i.$$

So in (1) there are equalities everywhere. Thus

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{ji} y_j \right) x_i \Rightarrow c_i x_i = \left(\sum_{j=1}^{m} A_{ji} y_j \right) x_i \text{ for } i = 1, \ldots n.$$

And finally for every $x_i,\ i = 1, \ldots n$:

$$x_i \neq 0 \quad c_i x_i = \left(\sum_{j=1}^{m} A_{ji} y_j \right) x_i \Rightarrow c_i = \left(\sum_{j=1}^{m} A_{ji} y_j \right).$$

⇐ Similarly to the previous part we know that:

$$x_i c_i = \left(\sum_{j=1}^{m} A_{ji} y_j \right) x_i \qquad \forall i = 1, \ldots, n,$$

$$y_j b_j = \left(\sum_{i=1}^{n} A_{ji} x_i \right) y_j \qquad \forall j = 1, \ldots, m.$$

Thus
$$\sum_{j=1}^{m} b_j y_j = \sum_{j=1}^{m} \sum_{i=1}^{n} A_{ji} x_i y_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{ji} y_j \right) x_i = \sum_{i=1}^{n} c_i x_i.$$
The above equality is equivalent to x, y being optimal solutions to the primal and the dual linear programs, respectively. Indeed for feasible solution $x^\star$ to the primal we have by weak duality
$$\sum_{i=1}^{n} c_i x_i^\star \geq \sum_{j=1}^{m} b_j y_j = \sum_{i=1}^{n} c_i x_i.$$
Thus x is an optimal solution to the primal program and similarly y is an optimal solution to the dual.

■

References

[1] Mateusz Golebiewski, Maciej Duleba: *Scribes of Lecture 5 in Topics in TCS 2015. http://theory.epfl.ch/courses/topicstcs/Lecture52015.pdf*