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Lecture 3: Matchings, Intro to Linear Programming

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In this lecture we do the following:

- We give the augmenting path algorithm for maximum cardinality (unweighted) bipartite matching.
- We introduce and define a powerful algorithmic method: linear programming.

For further reading about linear programming, I recommend the excellent text book *Understanding and Using Linear Programming* by Matousek and Gärtner available here:

https://link.springer.com/book/10.1007%2F978-3-540-30717-4

1 Algorithm for Maximum Cardinality Bipartite Matching

This section is taken from the lecture notes by Michel Goemans: *http://math.mit.edu/~goemans/18433S09/matching-notes.pdf*

1.1 Duality

Before giving an algorithm, perhaps an easier question is: how would you give a short proof that a given matching is optimal?

For this purpose, one would like to find upper bounds on the size of the largest matching and hope that the smallest of these upper bounds be equal to the size of the largest matching. This is a *duality* concept that will be very important in this subject. In this case, the dual problem will be a famous combinatorial optimization problem: vertex cover.

Vertex cover: A vertex cover is a set C of vertices so that all edges e of E are incident to at least one edge in C. In other words, there is no edge completely contained in $V \setminus C$.

We clearly have the following:

$$|M| \leq |C| \qquad \text{for any matching } M \text{ and vertex cover } C.$$

This follows from the fact that, given any matching M, a vertex cover C must contain at least one of the end points of each edge in M.

This is *weak duality*: The maximum size of a matching is at most the minimum size of a vertex cover. We shall in fact prove strong duality (that equality holds) for bipartite graphs:

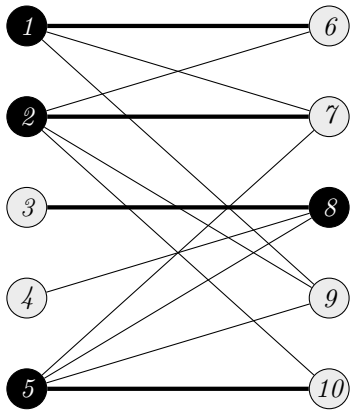
Theorem 1 (König 1931) *For any bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover.*

The proof of this theorem will be algorithmic. It will give an efficient algorithm for both finding a maximum size matching and a minimum size vertex cover of a bipartite graph. Note that whenever one has min max statement as above, the problem lies in $NP \cap coNP$ which may indicate that it has an efficient algorithm[2].

[1] **Disclaimer:** These notes were written as notes for the lecturer. They have not been peer-reviewed and may contain inconsistent notation, typos, and omit citations of relevant works.

[2] This sentence is hard to understand if you haven't taken a computational complexity course.

Example 1 *The edges* $(1,6),(2,7),(3,8)$ *and* $(5,10)$ *form a matching (of maximum cardinality). Vertices* $1,2,5$*, and* 8 *form a vertex cover (of minimum cardinality).*



1.2 Algorithm

Recall that a path is a collection of edges $(v_0,v_1),(v_1,v_2),\ldots,(v_{k-1},v_k)$ where the v_i's are distinct vertices. We can simply represent a path as $v_0 - v_1 - v_2 - \ldots - v_k$.

Definition 2 (Alternating path) *An* alternating path *with respect to* M *is a path that alternates between edges in* M *and edges in* $E \setminus M$*.*

Definition 3 (Augmenting path) *An* augmenting path *with respect to* M *is an alternating path in which the first and last vertices are unmatched.*

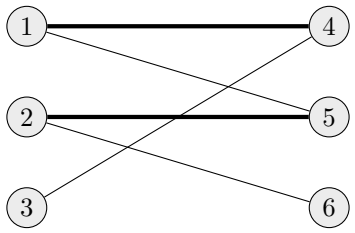


Figure 1: The edges $(3,4),(4,1),(1,5),(5,2),(2,6)$ form an augmenting path

The definition of an augmenting path motivates the following algorithm:

AugmentingPathAlgorithm(G):
Input: A bipartite graph $G=(V,E)$.
Output: A matching M of maximum cardinality.

1. Initialize $M=\emptyset$.
2. **while** exists an augmenting path P
3. update $M = M\Delta P \equiv (M \setminus P) \cup (P \setminus M)$.
4. **return** M.

Exercise 1 *Devise an efficient algorithm for finding an augmenting path* P *(if one exists). What is the total running time of the* AugmentingPathAlgorithm*?*

We now prove that AugmentingPathAlgorithm indeed finds a maximum matching.

Theorem 4 *A matching M is maximum if and only if there are no augmenting paths with respect to M.*

Proof (By contradiction)

($\Rightarrow$) Let P be some augmenting path with respect to M. Then $M' = M \Delta P$ is matching of greater cardinality than M. This contradicts the optimality of M.

($\Leftarrow$) If M is not maximum, let M^* be a maximum matching so that $|M^*| > |M|$. Let $Q = M \Delta M^*$. Then

- Q has more edges from M^* than from M (since $|M^*| > |M|$ implies that $|M^* \setminus M| > |M \setminus M^*|$).
- Each vertex is incident to at most one edge in $M \cap Q$ and one edge in $M^* \cap Q$.
- Thus Q is composed of cycles and paths that alternate between edges from M and M^*.
- Therefore there must be some path with more edges from M^* in it than from M. This path is an augmenting path with respect to M.

Hence, there must exist an augmenting path P with respect to M, which is a contradiction. ■

1.3 Proof of König's Theorem

Consider a maximum matching M^* of $G = (A \cup B, E)$. Let L the vertices reachable from unmatched vertices in A by alternating paths with respect to M^*. (So e.g. in the figure of Example 1 $L = \{4, 8, 3\}$.) The following proves König's theorem.

Lemma 5 *We have that $C^* = (A \setminus L) \cup (B \cap L)$ is a vertex cover. Moreover, $|C^*| = |M^*|$.*

Proof We first show that C^* is a vertex cover. Suppose toward contradiction that it is not. Then there is an edge $e = (a, b) \in E$ with $a \in A \cap L$ and $b \in B \setminus L$. The edge cannot belong to the matching. If it did then b should be in L because otherwise a would not be in L. Hence $e \in E \setminus M$. This however, implies that there is an alternating path from an unmatched vertex to b (namely go to a and take the edge (a, b)) contradicting the fact that $b \notin L$).

To show the second part of the proof, we show that $|C^*| \leq |M^*|$, since the reverse inequality is true for any matching and any vertex cover. The proof follows from the following observations:

1. No vertex in $A \setminus L$ is unmatched by the definition of L;
2. No vertex in $B \cap L$ is unmatched since this would imply the existence of an augmenting path (which contradicts that M^* is optimal).
3. There is no edge of the matching between a vertex $a \in A \setminus L$ and a vertex $b \in B \cap L$. Otherwise a would be in L.

These remarks imply that every vertex in C^* is matched and moreover the corresponding edges of the matching are distinct. Hence, $|C^*| \leq |M^*|$

■

2 Linear Programming

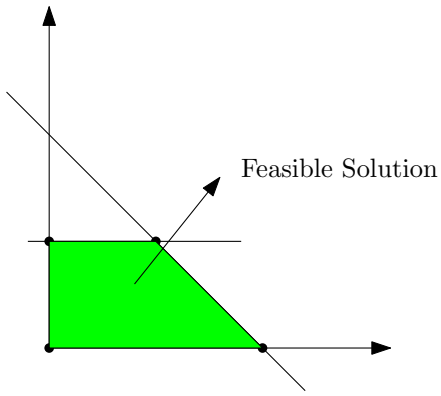
(Large parts of the following is taken from [1].)

A linear programing problem is the problem of finding values for variables that optimize a given linear objective function, subject to linear constraints.

Let us state an example of a linear program:

$$\begin{aligned} \text{Maximize} \quad & x + y \\ \text{Subject to} \quad & x + y \leq 2 \\ & y \leq 1 \\ & x, y \geq 0 \end{aligned}$$

The following figure shows the feasible area.



Definition 6 *A linear program (LP) is the problem of finding values for n variables $x_1, x_2, \ldots, x_n \in \mathbb{R}$ that minimize (or equivalently, maximize) a given linear objective function, subject to m linear constraints*

$$\begin{aligned} \textbf{minimize:} \quad & \sum_{i=1}^{n} c_i x_i & \\ \textbf{Subject to:} \quad & \sum_i e_{i,j} x_i = b_j & \text{for } j = 1, \ldots, m_1 \\ & \sum_i d_{i,k} x_i \geq g_k & \text{for } k = 1, \ldots, m_2 \\ & \sum_i f_{i,p} x_i \leq l_p & \text{for } p = 1, \ldots, m_3 \end{aligned}$$

where $m_1 + m_2 + m_3 = m$.

2.1 Motivation

Linear Programming is a very powerful tool - it can be used in many applications, both industrial and theoretical such as:

- obtaining an optimal production plan to maximize a profit of a factory
- modeling network flow problems (what is maximum possible flow that doesn't exceed capacity of each connection)
- theory - many theoretical problems can be introduced as Integer Programs(LP in which $x_i \in \mathbb{Z}$) that can be relaxed to LP obtaining a good approximation of optimal result- we'll see such applications later in the course.

2.2 Some history

It was first formalized and applied to problems in economics in the 1930s by Leonid Kantorovich. Kantorivich's work was hidden behind the Iron Curtain (where it was largely ignored) and therefore unknown in the West. Linear programming was rediscovered and applied to shipping problems in the early 1940s by Tjalling Koopmans.

Simplex Method was published by George Dantzig in 1947: it is the first complete algorithm to solve linear programming problems.

The principle is the following: we start from an extreme point and then we look at its neighbors. If one of these is better we move to it and continue in the same way, else we stop. Once we stop, we can be sure that we have an optimal solution, since we're in a convex polytope. Even if it's usually extremely fast, we know some bad examples where this method visits an exponential number of extreme points before to reach a solution, so this method does not always run in polynomial time.

Ellipsoid Method was studied by Leonid Khachiyan in the seventies.

This method is guaranteed to run in poly time (exactly $poly(n, m, \log(u))$ where u is the largest constant) but it is slow in practice.

We do a binary search for the optimal value of the objective function, so we can add the objective function as a constraint, and just need to decide if there exists a feasible point. We start by taking an ellipsoid surrounding the feasible area. We then check the center of the ellipsoid, if it's inside the area then we have our solution, else we identify a violated constraint, and cut our ellipsoid in two parts using this constraint, and construct a new ellipsoid around the part of the old ellipsoid in which the constraint is satisfied. We repeat this process until we find a feasible point or can be sure that the feasible area is empty.

Interior point Method developed by Narendra Karmarkar in 1984. In this method we move in the feasible region to find an OPT solution.

When solving discrete optimization problems (such as matchings, vertex cover, spanning tree, etc.) it will be important that optimal solutions have certain structure. That is where we use extreme points and their structure (which is also important for the simplex algorithm).

References

[1] Ashkan Norouzi-Fard, Christos Kalaitzis: *Scribes of Lecture 9 in Topics in TCS 2014. http://theory.epfl.ch/courses/topicstcs/Lecture9.pdf*